Probability of Winning the Lottery

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Overview

During this unit, students will investigate a real-world application of probability by determining the probability of winning the lottery. Students will be introduced to the counting principle, permutations, and combinations, which will later be used to calculate one's chances of winning various payout amounts in the state lottery game. Students will also conduct a computer simulation of the lottery game to determine if any patterns or trends arise. This activity will also be supplemented by a reading in psychological research on our inclination to observe patterns in our environment, even if they do not exist.

Rationale

The jackpot payout for the lottery continues to grow as the new Mega Millions and the Powerball lottery games gain popularity. It is now possible for an individual to win millions of dollars by correctly picking a lineup of 6 or 7 numbers from a set of approximately 50 numbers. The overall chances of winning a prize are very small; however, many are willing to risk the $1 ticket price in order for a shot at the grand prize. This unit will look at the statistical probability of winning money from playing the lottery to identify if trends or patterns exist in lottery results.

Objectives

This unit is intended to help high school mathematics students to apply probabilities to a real world application: the lottery.

The objectives of the unit will include the following:

- Apply the counting principle to determine the chances of winning a simple 4-number lottery game.
- Apply the counting principle to determine the chances of winning different payouts, including the jackpot, in a traditional 6-number lottery game.
- Explore different strategies for designing a lottery game.
- Identify any patterns or trends in results from a computer simulation.

Strategies
This multi-day, project-based unit will include hands-on activities in order for students to better understand and visualize the probability of winning in each lottery game. Students will also compare the theoretical and experimental probabilities of winning by playing online, free versions of various lottery games, providing by the Pennsylvania lottery website.

Standards
The Core Curriculum of the School District of Philadelphia is aligned to the Pennsylvania Academic Standards for Mathematics. This unit will include instruction that meets the following standards:

11.E.3.1: Apply probability and/or odds to practical situations.

- Find probability for independent, dependent, or compound events and represent as a fraction, decimal, or percent.
- Find, convert, and/or compare the probability and/or odds of a simple event.

11.E.3.2: Apply counting techniques in problem-solving settings.
Determine the number of permutations and/or combinations or apply the fundamental counting principle.

Classroom Activities

Lesson 1: The Counting Principle

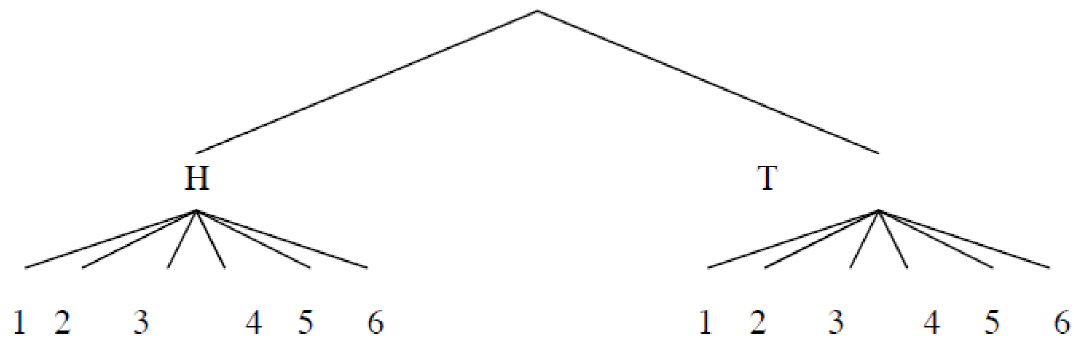
The counting principle is used to determine the number of possible outcomes for a sequence of independent events. The term "independent" indicates that the events should not depend on the order in which they are completed.

For example, let's say that we have two events, E_1 and E_2, where E_1 can happen in n_1 different ways and E_2 can happen in n_2 different ways. There are n_1 x n_2 different possible outcomes for these events to occur.

We can extend this example to any number of events by using the counting principle. Let's now say that we have k events (E_1, E_2, ... E_k). The number of possible outcomes for each event corresponds to n_1, n_2, … n_k. The total number of possible outcomes for k events is n_1 x n_2 x … x n_k.

Example 1: A coin is tossed and a 6-sided die is rolled. Find the number of outcomes for the sequence of events.

Solution: For this problem, it may help to draw a tree diagram to illustrate the number of possible outcomes. A coin has 2 possible outcomes: heads or tails. A 6-sided die has 6 possible outcomes: 1, 2, 3, 4, 5, or 6.



By examining the tree diagram, it is clear that there are 12 different possible outcomes. We can come to this conclusion by multiplying the number of outcomes for each event: 2 x 6.

Example 2: For her vacation, Angela packed 5 shirts, 4 pants, and 2 shoes. How many possible outfits can Angela create?

Solution: Total number of outcomes = 5 x 4 x 2 = 40
She can create 40 different outfits with the clothes she packed for the vacation.

Lesson 1: Exercises

1. Michael wants to create a sandwich, and he has the following options for bread, meat, and cheese:

   Bread: White or Wheat
   Meat: Turkey, Ham, Roast Beef
   Cheese: American, Swiss, Provolone

   Draw a tree diagram to illustrate the possible sandwiches that Michael can make. How many possible sandwiches can he make from those ingredients?

2. The digits 0, 1, 2, 3, and 4 are to be used in a 4-digit ID card. How many different cards are possible if digits may be repeated?

3. For a banquet, a committee can select beef, pork, chicken, or veal; baked potatoes or mashed potatoes; and peas or green beans for a vegetable.

   Draw a tree diagram for all possible choices of meat, potatoes, and vegetables. How many possible outcomes are there?

4. How many possible outcomes are there if three coins are tossed?

5. Next year you are taking Algebra II, English, American History, Chemistry, Spanish, and Music. Each class is offered during each of the six periods in the day. In how many different possible ways can you schedule your classes?

Lesson 2: Permutations

A permutation is the number of ways that a set of objects may be arranged in a particular order.

The Permutation Rule may be used to calculate the number of ways of selecting r objects from a set of n objects. The notation for permutation is P(n, r) or ${}_nP_r$.

$${}_nP_r = n! / (n - r)!$$

Example 1: In how many different ways can the letters in the word CAT be rearranged?

Solution: There are 6 permutations: CAT, CTA, ACT, ATC, TCA, TAC.

Although this example was quite simple, it is useful to be able to determine the number of permutations without listing them for more complex problems.

Example 2: The general manager of a fast-food restaurant chain must select 4 restaurants from 11 for a promotional program. How many different possible ways can the selection be done?

Solution: Rather than listing each permutation, we may use the counting principle:

There are 11 ways to fill the first position.
There are 10 ways to fill the second position (1 restaurant has already been selected).
There are 9 ways to fill the third position (2 restaurants have already been selected).
There are 8 ways to fill the fourth position (3 restaurants have already been selected).

In this way, the number of permutations is 11 x 10 x 9 x 8 = 11! / 7! = 7,920

Lesson 2: Exercises

1. How many different 4-letter permutations can be formed from the letters in the word *decagon*?

2. At a local cheerleading camp, 5 routines must be practiced. A routine may not be repeated. In how many different orders can these 5 routines be presented?

3. How many 5-digit zip codes are possible if digits can be repeated? How many 5-digit zip codes are possible if digits cannot be repeated?

Lesson 3: Combinations

A combination is the number of ways that a set of objects may be arranged in no particular order.

The Combination Rule may be used to calculate the number of ways of selecting r objects from a set of n objects. The notation for combination is C(n, r) or ${}_nC_r$.

$${}_nC_r = n! / r!(n - r)!$$

Example 1: A bicycle shop owner has 12 mountain bicycles in the showroom. The owner wishes to select 5 of them to display at a bicycle show. In how many different ways can a group of 5 be selected?

Solution: Rather than listing each combination, we may use the counting principle:

There are 12 ways to fill the first position.
There are 11 ways to fill the second position (1 bicycle has already been selected).
There are 10 ways to fill the third position (2 bicycles have already been selected).
There are 9 ways to fill the fourth position (3 bicycles have already been selected).
There are 8 ways to fill the fifth position (4 bicycles have already been selected).

However, we must remember that since the order of the bicycles does not matter, we need to divide by the number of ways that the 5 bicycles can be rearranged (5! or 120).

In this way, the number of combinations is 12 x 11 x 10 x 9 x 8 / 120 = 12! / 7!5! = 792

Example 2: There are 7 women and 5 men in a club. A committee of 3 women and 2 men needs to be chosen. In how many different ways can the committee be selected?

Solution: We need to consider the women and the men separately for this problem.

For the women, n = 7 and r = 3. The number of combinations is: 7 x 6 x 5 / 6 = 35.
For the men, n = 5 and r = 2. The number of combinations is: 5 x 4 /2 = 10.
The total number of combinations is: 35 x 10 = 350.

Lesson 3: Exercises

1. The summer Olympics games had 16 countries qualify to compete in soccer. In how many different ways can teams of 2 be selected?

2. In how many ways can a jury of 6 women and 6 men be selected from 10 women and 12 men?

Lesson 4: Probabilities in Big 4 Lottery Game

The Pennsylvania state lottery offers several variations of lottery games. In one of the simplest games, Big 4, a player picks any four-digit number and places a bet ranging from $0.50 to $5.00. The player wins if his or her number is selected in the daily drawing.

The payouts for lottery games are determined by one's odds of winning. The odds in favor of an event are typically expressed as a ratio of the likelihood that an event will happen to the likelihood that an event will not happen.

Activity:
There are several different betting strategies for Big 4 listed below. Students should calculate the probability of winning the Big 4 lottery for each strategy using the counting principle.

1. *Play It Straight:* Player plays 4 different digits. Player wins only with exact match.
2. *Play It Boxed:* Player plays 3 of the same digit and 1 other digit. Player wins if the number is drawn in any order.
3. *Box 2 Pairs:* Player picks 2 pairs of numbers. Player wins if the number is drawn in any order.
4. *Box 1 Pair + 2 Digits:* Player picks 1 pair of number and 2 other digits. Player wins if the number is drawn in any order.
5. *Box 4 Different Digits:* Player plays 4 different digits. Player wins if the number is drawn in any order.

Students may play a game demo of the Big 4 lottery game at the website: http://www.palottery.state.pa.us/Game_Demos/big_4/index.html.

Solutions:
In order to calculate the odds of winning, one must know the total number of different 4-digit numbers. There are 10 ways to pick the first number, 10 ways to pick the second number, 10 ways to pick the third number, and 10 ways to pick the fourth number.

The number of different 4-digit numbers is 10 x 10 x 10 x 10 = 10,000.

1. *Play It Straight: 1/10,000*
   Since the order of the numbers matters, there is exactly 1 chance to win.
   The number may only be in the order: abcd.

2. *Play It Boxed: 4/10,000*
   Since the number may be drawn in any order, there are 4 chances to win.
   The numbers may be in any of the following orders: abbb, babb, bbab, or bbba.

3. *Box 2 Pairs: 6/10,000*
   Since the number may be drawn in any order, there are 6 chances to win.
   The numbers may be in any of the following orders: aabb, bbaa, abab, baba, abba, or baab.

4. *Box 1 Pair + 2 Digits: 12/10,000*
   Since the number may be drawn in any order, there are 12 chances to win.
   The numbers may be in any of the following orders: aabc, aacb, bcaa, cbaa, abca, acba, baac, caab, abac, acab, baca, caba.

5. *Box 4 Different Digits: 24/10,000*
   Since the number may be drawn in any order, there are 24 chances to win.
   The order of the 4 numbers does not matter, so the total number of permutations is 4 x 3 x 2 x 1 = 24.

<u>Lesson 5: Big 2 Lottery Game Simulation</u>

Experimental probability describes the chance of an event occurring, based on repeated testing and observing the results. Mathematically, it is the ratio of the number of times an event occurred to the total number of trials. In practice, one may find the experimental probability of winning a game by dividing the number of games won by the total number of games played.

Theoretical probability describes the chance of an event occurring under ideal circumstances. For example, the probability of rolling a 4 on a six-sided die is 1/6.

In Lesson 4, students calculated the theoretical probability of winning the Big 4 lottery game with various betting strategies. In this lesson, students will use a computer simulator to calculate the probability of winning in a set number of drawings; however, the simulation activity will only use 2 numbers, instead of 4. The simulator randomly selects an integer between 1 and 99, which represents the winning lottery combination for that particular drawing.

The source code for the computer simulator is listed in the Appendix.

Activity:
Students should open the Big 2 lottery simulation website on their computers. Using the following worksheet, students should record the first 25 lottery picks.

In their analysis, they will need to determine how many times they won. Students will compare theoretical probability versus experimental probability and will write a paragraph describing their results.

Data Collection Sheet:
Using the simulator, select 25 lottery numbers and record them below.

1. What is the experimental probability that a player will win?
2. Were your theoretical results exactly the same as your experimental results?
3. If no, why do you think your results differed from the theoretical probabilities?

<u>Lesson 6: Expected Value</u>

Expected value is the most likely value of a random variable. In the case of an investment decision, it is the average value of all possible payoffs.

In order to find the expected value of a variable, you must first multiply each possible payoff by its probability of occurring, followed by adding all of the products together.

For example, let's say that you roll a 6-sided die. If you roll a 3, then you win $5.00. If you don't roll a 3, then you have to pay $1.00. What is the expected value of the game?

You should recognize that since the probability of rolling a 3 is 1/6, the probability of not rolling a 3 is 5/6. Therefore, expected value = P(3) • (5) + P(not 3) • (-1) = (1/6) • (5) + (5/6) • (-1) = 5/6 - 5/6 = 0.

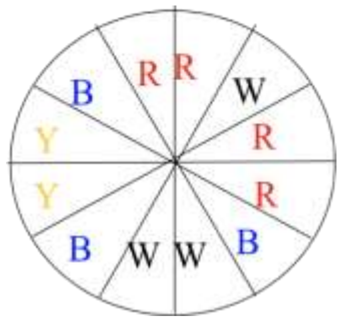
If the expected value is 0, we say the game is fair.

Lesson 6: Exercises

Find the expected value of the following games:
1. Roll another die. If you roll a 3 or a 5, you win 25¢. If you roll a 1, you get $1.00. If you roll an even number, you pay 50¢.

2. There are four choices on a multiple-choice question. If you get the right answer, you earn a point. If you get the wrong answer, you lose a point. Is this grading system fair?

3. You spin the spinner below. If you spin blue or white, you get a quarter. If you spin red, you get a nickel. If you spin yellow, you have to pay 1 dollar.



Lesson 7: Expected Value of the Lottery

Activity:
In a certain state lottery, a player chooses three digits, which must be in a specific order. Note that the numbers may lead with the digit 0, so numbers such as 056 or 009 are acceptable numbers. Digits may also be repeated.

In each lottery drawing, a three-digit sequence is selected. Any player with a lottery pick matching all three digits, in the correct order, receives a payout of $500.

1. Determine the probability of winning this lottery game.
2. Calculate the expected value of winning if its costs $3 to play one game.
3. Determine a fair cost for an individual to play this lottery game.

Solutions:
1. Let's say that we chose the number 2, 9, and 2, in that order. On the first draw, the probability of drawing a 2 out of 10 possible numbers is 1/10. Because digits may be repeated, the 2 is replaced. On the second draw, the probability of now drawing a 9 is still 1/10. The 9 is then replaced. On the third draw, the probability of drawing a 2 is still 1/10.

   We can now multiply the probabilities together to determine the overall probability of winning: 1/10 x 1/10 x 1/10 = 1/1000

2. The expected value of winning is: P(winning) • $497 + P(not winning) • $-3 = (1/1000) • (497) + (999/1000) • (-3) = 0.497 – 2.997 = -2.5
   In other words, you are expected to lose the game.

3. The expected value for a fair game should be 0. The probability of winning the game is still 1/1000.

   We can set the equation for expected value equal to 0 to solve for a fair cost of the game: (1/1000) • (500 - x) + (999/1000) • (-x) = 0
   This simplifies to: 0.5 – 0.001x – 0.999x = 0, which may be further simplified to 0.5 – x = 0
   The above equation may be solved to produce a value of 0.50. This means that the game must cost 50 cents to play in order for it to be a fair game!

Lesson 8: Mega Millions

Mega Millions is a national lottery game that is known for its large rollover jackpot payouts. Each player picks 6 total numbers: 5 different numbers from 1 to 56 (the "white numbers") and 1 number from 1 to 46 (the "Mega number"). A player wins the jackpot if he or she matches all six numbers selected in a drawing.

Activity:

1. Use the counting principles to calculate the probability of winning the Mega Millions jackpot.
2. Use the counting principles to calculate the probabilities of other winning combinations.
3. Calculate the expected value of the game to determine if it is worth purchasing a $1 lottery ticket for a chance to win the large jackpot prize. Use a jackpot value of $42 million for your calculation.

Solutions:

1. The numbers of ways to select 5 numbers from a pool of 56 numbers may be calculated using combinations: ${}_{56}C_5$ = 3,819,816. The number of ways to select 1 number from a pool of 46 numbers is 46. Thus, the total number of Mega Millions combinations is 3,819,816 x 46 = 175,711,536.

   There is only one way that the first five numbers on your lottery ticket can match the five selected white numbers. There is also only one way for the sixth number on your lottery ticket to match the Mega Number. Therefore, there is one way to win the jackpot.

   The probability of winning the jackpot is 1/175,711,536.

2. The probabilities of the other winning combinations are:
   a. **Match all 5 white numbers but not the Mega number (payout = $250,000):** There is only one way that the first five numbers on your lottery ticket can match the five selected white numbers (${}_5C_5$). There are 45 ways for your sixth number to match any of the 45 losing Mega

numbers ($_{45}C_1$). The number of ways to achieve this combination is 1 x 45 = 45, which leads to a probability of 45/175,711,536. This simplifies to about "one chance in 3,904,701."

b. **Match 4 out of 5 white numbers and the Mega number (payout = $10,000):** There are five ways that four of the first five numbers on your lottery ticket can match the five selected white numbers ($_5C_4$). There are 51 ways for your fifth white number to match any of the 51 losing white numbers ($_{51}C_1$). There is one way for your sixth number to match the winning Mega number ($_1C_1$). The number of ways to achieve this combination is 5 x 51 x 1 = 255, which leads to a probability of 255/175,711,536. This simplifies to about "one chance in 689,065."

c. **Match 4 out of 5 white numbers but not the Mega number (payout = $150):** There are five ways that four of the first five numbers on your lottery ticket can match the five selected white numbers ($_5C_4$). There are 51 ways for your fifth white number to match any of the 51 losing white numbers ($_{51}C_1$). There are 45 ways for your sixth number to match any of the 45 losing Mega numbers ($_{45}C_1$). The number of ways to achieve this combination is 5 x 51 x 45 = 11,475, which leads to a probability of 11,475/175,711,536. This simplifies to about "one chance in 15,313."

d. **Match 3 out of 5 white numbers and the Mega number (payout = $150):** There are ten ways that three of the first five numbers on your lottery ticket can match the five selected white numbers ($_5C_3$). There are 1,275 ways for two of your white numbers to match any of the 51 losing white numbers ($_{51}C_2$). There is one way for your sixth number to match the winning Mega number ($_1C_1$). The number of ways to achieve this combination is 10 x 1,275 x 1 = 12,750, which leads to a probability of 12,750/175,711,536. This simplifies to about "one chance in 13,781."

e. **Match 3 out of 5 white numbers but not the Mega number (payout = $7):** There are ten ways that three of the first five numbers on your lottery ticket can match the five selected white numbers ($_5C_3$). There are 1,275 ways for two of your white numbers to match any of the 51 losing white numbers ($_{51}C_2$). There are 45 ways for your sixth number to match any of the 45 losing Mega numbers ($_{45}C_1$). The number of ways to achieve this combination is 10 x 1,275 x 45 = 573,750, which leads to a probability of 573,750/175,711,536. This simplifies to about "one chance in 306."

f. **Match 2 out of 5 white numbers and the Mega number (payout = $10):** There are ten ways that two of the first five numbers on your lottery ticket can match the five selected white numbers ($_5C_2$). There are 20,825 ways for three of your white numbers to match any of the 51 losing white numbers ($_{51}C_3$). There is one way for your sixth number to match the winning Mega number ($_1C_1$). The number of ways to achieve this combination is 10 x 20,825 x 1 = 208,250, which leads to a probability of 208,250/175,711,536. This simplifies to about "one chance in 844."

g. **Match 1 out of 5 white numbers and the Mega number (payout = $3):** There are five ways that one of the first five numbers on your lottery ticket can match the five selected white numbers ($_5C_1$). There are 249,900 ways for three of your white numbers to match any of the 51 losing white numbers ($_{51}C_4$). There is one way for your sixth number to match the winning Mega number ($_1C_1$). The number of ways to achieve this combination is 5 x 249,900 x 1 = 1,249,500, which leads to a probability of 1,249,500/175,711,536. This simplifies to about "one chance in 141."

h. **Match 0 out of 5 white numbers and the Mega number (payout = $2):** There is one way that none of the first five numbers on your lottery ticket can match the five selected white numbers ($_5C_0$). There are 2,349,060 ways for five of your white numbers to match any of the 51 losing white numbers ($_{51}C_5$). There is one way for your sixth number to match the winning Mega number ($_1C_1$). The number of ways to achieve this combination is 1 x 2,349,060 x 1 = 2,349,060, which leads to a probability of 2,349,060/175,711,536. This simplifies to about "one chance in 75."

3. The expected value of winning is:
   P(jackpot) • $42,000,000 +
   P(match 5 white, not Mega) • $250,000 +
   P(match 4 white and Mega) • $10,000 +
   P(match 4 white, not Mega) • $150 +
   P(match 3 white and Mega) • $150 +
   P(match 3 white, not Mega) • $7 +
   P(match 2 white and Mega) • $10 +
   P(match 1 white and Mega) • $3 +
   P(match 0 white and Mega) • $2
   P(not winning) • $-1 =
   -0.55 (In other words, you are expected to lose in this game.)

Lesson 9: Design a Lottery Game

Individuals who design lottery games must consider a variety of factors in order to ensure that players have a decent chance of winning and that the house will likely make a profit. As a result, it is important for lottery games to be carefully designed so that they are not too easy (where the house would not profit) or too hard (where there is little incentive to play).

Activity:

1. Students should read the first page of a research paper published in the *Journal of the Operational Research Society*. The paper discusses various factors that must be considered in the design of lottery games, including the incentive of a rollover jackpot. The first page is available at: http://www.jstor.org/pss/822753.

2. For one of the final projects of the unit, students will serve as an advisor to a state-sponsored lottery game. They will need to design the rules to the game, decide payoff amounts for winning, and calculate the odds for winning.

3. For extra credit, students may have other individuals play their lottery game (this is strictly theoretical to avoid the use of real money). They should keep track of winners and the payoff amounts for each win. At the end, they should calculate the profit for the house and reflect on the design of the game. If necessary, they may suggest modifications to the lottery game that would make it more successful.

Lesson 10: Psychological Research

According to an article in the journal *Scientific American*, it is human inclination to observe patterns in the environment, even if they do not exist.

Activity:

1. Students should look at their results from the lottery simulation game in Lesson 5 and try to identify if any patterns exist in the results.

2. Students should read and summarize the article in the journal *Scientific American* that discusses the tendency for people to see patterns when none exist in attempt to explain unpredictable situations. The first page is available at: http://www.scientificamerican.com/article.cfm?id=patternicity-finding-meaningful-patterns.

3. Students should write a one-page reflection on why individuals might try to identify patterns in a variety of situations, including the lottery and the stock market.

Bibliography

Teacher Resources

1. Bluman, Allan G. Elementary Statistics: A Step-By-Step Approach. New York: McGraw-Hill, 2007.

2. Butler, Bill. Mega Million Odds. 1 June 2011. <http://www.durangobill.com/MegaMillionsOdds.html>.

Student Resources

3. Hartley R. and Lanot G. "On the Design of Lottery Games." Journal of the Operational Research Society. 54 (2003): 89. 19 June 2011 <http://www.jstor.org/pss/822753>.

4. Pennsylvania Lottery. Big 4. 15 Apr. 2011 <http://www.palottery.state.pa.us/games.aspx?id=434>.

5. Shermer, Michael. "Patternicity: Finding Meaningful Patterns in Meaningless Noise." Scientific American. 25 Nov. 2008. 19 June 2011 <http://www.scientificamerican.com/article.cfm?id=patternicity-finding-meaningful-patterns>.

Appendix:

This is the source code for the Big 2 lottery simulation website:

```
<html>
<head>
<title>Lottery Simulation</title>
</head>
<script language="JavaScript">

//t is where I keep the list of lottery drawings so far
var t=""
var N=0

function draw(){
 var x1=Math.floor(10*Math.random())
 var x2=Math.floor(10*Math.random())
 var x3=Math.floor(10*Math.random())
 var x4=Math.floor(10*Math.random())
 var y1,y2,y3,y4

y1=document.forms[0].a1.value
y2=document.forms[0].a2.value
//y3=document.forms[0].a3.value
```

```
//y4=document.forms[0].a4.value

dw.innerHTML="Drawing: "+x1+" "+x2
 N=N+1
 if(N==11){N=0
 t=t+"<p>"
 }

if(x1==y1 && x2==y2){
 last.innerHTML="Drawing: "+x1+" "+x2+"<p>You win straight!"
 t=t+"W"
 }else{
  if(x2==y1 && x1==y2){
  last.innerHTML="Drawing: "+x1+" "+x2+"<p>You win boxed!"
  t=t+"B"
  }else{
   last.innerHTML="Drawing: "+x1+" "+x2+"<p>You lose."
   t=t+"L"
  }
 }
//var a=x1+x2+x3+x4
//if(t==""){t=t+a}else{t=t+", "+a}
lst.innerHTML=t
//f1.innerHTML=a
}

</script>

<body bgcolor="white" text="navy">
<form>
<p> Pick two numbers, zero through nine:
<input name="a1" size=3>  <input name="a2" size=3>
<p>  Push the button for a new lottery drawing
<p>
<input type=button value="Pick Numbers" onClick="draw()">
<p>
<font id="dw"></font>
<p> <font id="last">
</font>
<p> So far:
<p> <font id="lst"> Nothing</font>
<p>
</form>
</body>
</html>
```